The Stationary Phase Error Distribution of a Digital Phase-Locked Loop

Glen Skiller and Dawei Huang

***Abstract*—The stationary properties of a first-order digital phase-locked loop based on the extended Kalman filter (EKF-PLL) are investigated. A discrete Markov chain approximation of the phase error process is used to derive the asymptotic distribution of the phase error as well as the distribution of the time at which the EKF-PLL is first "out of lock," given that it is in a steady state to begin with. These approximations are compared with large computer simulations.**

***Index Terms*—Discrete Markov chain, phase-locked loop, stationary distribution, time "out of lock".**

I. Introduction

THE phase-locked loop (PLL) has been an integral tool in the area of communications for several decades. Basically a feedback device for tracking the phase of an oscillating signal, it has been widely studied and comes in many forms, including the so-called digital PLL (DPLL) and software PLL [2]. Here, we consider a popular version of the PLL derived from the extended Kalman filter (EKF) [1], referred to as the EKF-PLL.

A good indication of a PLL's performance is given by the limit probability density function (pdf) of its phase error. A closed form of this function for an analog PLL in the fixed-frequency case was derived in [9]. However, no exact closed form of this function for a DPLL has been found. Methods of approximating the limit pdf have been proposed in [11], [4], [10], and [6], where types of DPLL known as the zero-crossing DPLL (ZCDPLL) and digital tanlock loop (DTL) were considered. In [7], quantization effects in a ZCDPLL were taken into account allowing the limit pdf of the phase error to be derived using discrete Markov chain (DMC) techniques. The ZCDPLL and DTL are based on nonuniform sampling rates, and the authors of these works considered only a constant or linearly changing phase.

The EKF-PLL considered here operates on a uniformly sampled signal with random walk phase. We present a novel method for approximating the limit pdf as well as the distribution of the first time at which the EKF-PLL is "out of lock" given that it is in a steady state to begin with. The method of calculation is based on approximating the phase error process by a DMC. This method is similar to that in [7], where the mean time to an "out-of-lock" state was also calculated, but in that case the discretization was intrinsic to the PLL and the resulting phase error difference equation. In our case, the phase error is always a continuous random variable, and we only discretize certain functions operating on this variable.

The signal under consideration is of the form

$$\begin{aligned} x_t &= x_{t-1} + u_t, u_t \sim N(0, \sigma_u^2) \\ y_t &= A \begin{bmatrix} \sin x_t \\ \cos x_t \end{bmatrix} + v_t, v_t \sim N(0, \sigma_v^2 I) \end{aligned} \quad (1)$$

where y_t is the observed signal at time t and x_t is the sequence being tracked by the PLL. Since the sampling rate can be chosen so that x_t is slowly varying in most applications and because of bandwidth constraints in communications, we will assume that $\sigma_u^2 \ll 1$. Also, we can assume that $\sigma_v^2 = 1$ since the quantity of importance is the signal-to-noise ratio (SNR) and this can be varied through A. For $\sigma_v^2 = 1$, the logarithmic SNR (in decibels) is given by $\mathrm{SNR} = 10 \log_{10}(A^2/2)$.

The particular PLL to be investigated is the limit EKF corresponding to model (1), which was derived in [1] and is given by

$$\hat{x}_t = \hat{x}_{t-1} + K[\cos \hat{x}_{t-1} \quad -\sin \hat{x}_{t-1}] y_t$$

where $K = 2\sigma_u / (\sqrt{A^2 \sigma_u^2 + 4} + A\sigma_u)$.

The method of computing the approximate limit pdf is discussed in the next section. In Section III, the time until the EKF-PLL is "out of lock" is discussed and equations for its approximate distribution are given. Section IV contains examples and comparisons of our results with simulated approximations. Section V contains a summary of the work.

II. Limit PDF of the Phase Error and Its Supporting Region

Theoretically, the unrestricted phase error process $\tilde{x}_t' = \hat{x}_t - x_t$ is not ergodic, so a unique stationary pdf will not exist. For this reason, we consider the process $\tilde{x}_t = \tilde{x}_t' \mathrm{mod} 2\pi$, which is restricted to the range $[-\pi, \pi]$. In practice, however, when σ_u^2 is small and the SNR is large enough, the properties of both processes are the same.

Under the assumption that $\sigma_u^2 \ll 1$, the transition pdf of $\tilde{x}_t$ can be shown to be given approximately by [8]

$$\begin{aligned} p(x|y) &= \Pr\{\tilde{x}_t = x | \tilde{x}_{t-1} = y\} \\ &\approx \sum_{k=-1}^{1} G[x; y - L\sin y - 2k\pi, (1 - L\cos y)^2 \sigma_u^2 + K^2] \end{aligned} \quad (2)$$

for x, $y \in [-\pi, \pi]$, where $L = AK \leq 1$ and $G(x; \mu, \sigma)$ denotes the Gaussian density with mean μ and variance σ^2. Since $p(x|y)$ is a strictly positive regular kernel on a bounded closed interval, a unique limit pdf for $\tilde{x}_t$ exists, it is independent of the initial

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The authors are with the School of Mathematics, Queensland University of Technology, Brisbane, Qld. 4001, Australia (e-mail: d.huang@fsc.qut.edu.au).

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distribution, and it is continuous on $[-\pi, \pi]$ ([5, Theorem 1, p. 265]). This limit pdf is the solution $\pi(x)$ to the steady-state Chapman–Kolmogorov equation

$$\pi(x) = \int_{-\pi}^{\pi} p(x|y)\pi(y)\, dy. \tag{3}$$

The aim of our method is to approximate the quantities

$$\pi_{N,n} = \int_{c_n}^{c_{n+1}} \pi(x)\, dx, \qquad n = 0, 1, \cdots, N-1$$

for some choice of $\{c_n, n = 0, 1, \cdots, N\}$. It can be shown [8] that $\pi(x)$ is symmetric about 0, so we only need to consider approximating one side. Also, values of $\pi(x)$ toward the tail, i.e., toward $x = \pm\pi$, can be negligible, so we can assume that $\pi(x) = 0$ whenever $|x| > C$ for some $C \le \pi$. We therefore choose $c_n, n = 0, 1, \cdots, N$ to satisfy

$$c_0 \le 0 < c_1 < \cdots < c_N = C \le \pi.$$

Obviously, as we increase N and choose $c_n, n = 0, 1, \cdots, N$ so that $\max_n(c_n - c_{n-1}) \to 0$ as $N \to \infty$, the set of values $\{\pi_{N,n}/(c_{n+1} - c_n), n = 0, 1, \cdots, N-1\}$ will approach the actual function $\pi(x)$.

A procedure for choosing C and $\{c_n, n = 0, 1, \cdots, N\}$ and calculating the approximation $\hat{\Pi}_N = [\hat{\pi}_{N,0}\ \hat{\pi}_{N,1} \cdots \hat{\pi}_{N,N-1}]^T$ for a given value of N is outlined as follows.

Start with $C = \pi$.

Step 1) Form the partition

$$c_n = \frac{2n-1}{2N-1}C, \qquad n = 0, 1, \cdots, N.$$

Let

$$\begin{aligned} d_n &= \frac{c_n + c_{n+1}}{2} \\ d_{-n} &= -d_n, \qquad n = 0, 1, \cdots, N-1 \end{aligned}$$

and

$$\hat{p}_{jk} = \frac{p(d_j|d_k)}{p(0|d_k) + \sum_{l=1}^{N-1} [p(d_l|d_k) + p(d_l|d_{-k})]},$$
$$j = 0, 1, \cdots, N-1;\ k = 1-N, \cdots, 0, \cdots, N-1.$$

Set

$$n = 1 \quad \hat{Q}_1 = 1 \quad \text{and} \quad \hat{\Pi}_1 = 1.$$

Step 2) For each $n \ge 2$, let

$$\begin{aligned} \alpha_n &= [2 \quad \hat{p}_{1,n-1} + \hat{p}_{1,1-n} \quad \hat{p}_{2,n-1} + \hat{p}_{2,1-n} \\ &\qquad \cdots \quad \hat{p}_{n-2,n-1} + \hat{p}_{n-2,1-n}]' \\ \beta_n &= [\hat{p}_{n-1,0} \quad \hat{p}_{n-1,1} + \hat{p}_{n-1,-1} \quad \hat{p}_{n-1,2} + \hat{p}_{n-1,-2} \\ &\qquad \cdots \quad \hat{p}_{n-1,n-2} + \hat{p}_{n-1,2-n}] \\ \gamma_n &= \hat{p}_{n-1,n-1} + \hat{p}_{n-1,1-n} - 1 \\ \delta_n &= \beta_n \hat{\Pi}_{n-1}. \end{aligned}$$

Then

$$\begin{aligned} \hat{Q}_n^{-1} &= \begin{bmatrix} I & -\hat{Q}_{n-1}{}^{-1}\alpha_n \\ 0 & 1 \end{bmatrix} \\ &\cdot \begin{bmatrix} \hat{Q}_{n-1}{}^{-1} & 0 \\ 0 & (\gamma_n - \beta_n \hat{Q}_{n-1}{}^{-1}\alpha_n)^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ -\beta_n \hat{Q}_{n-1}{}^{-1} & 1 \end{bmatrix} \end{aligned}$$

and

$$\hat{\Pi}_n = \begin{bmatrix} \hat{\pi}_{n,0} \\ \hat{\pi}_{n,1} \\ \vdots \\ \hat{\pi}_{n,n-1} \end{bmatrix} = \begin{bmatrix} \hat{\Pi}_{n1} \\ 0 \end{bmatrix} - \hat{Q}_n{}^{-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \delta_n \end{bmatrix}.$$

Step 3) If:

a) $n < N$ and $\max\{\hat{\pi}_{n,n-1}, |\hat{\pi}_{n,j} - \hat{\pi}_{n-1,j}|, j = 0, 1, \cdots, n-2\} < (2C/2N-1)\epsilon$ for some small ϵ, let $C = c_n = (2n-1/2N-1)C$ and go to Step 1), or

b) $n = N$, the algorithm is finished.

Otherwise, let $n = n+1$ and repeat Step 2).

It can be shown using the theory of finite elements (see, for example, [3, Theorem 4.4.20, p. 107]) that $\hat{\Pi}_N \to \Pi_N$ as $N \to \infty$, so the approximation approaches the true function $\pi(x)$ as N increases.

III. The First Time "Out of Lock"

We can use our method of approximation to obtain other information about the EKF-PLL such as the distribution of T, the length of time the EKF-PLL runs until it becomes "out of lock," given that it begins in a steady state. A PLL is said to be "out of lock" when

$$|\tilde{x}_t'| = |\hat{x}_t - x_t| > \pi.$$

As mentioned previously, when $\sigma_u^2 \ll 1$ and the SNR is not too low, we can investigate $\tilde{x}_t$ instead of $\tilde{x}_t'$. We define T to be smallest value of t for which

$$|\tilde{x}_t| > c_{N-1}$$

where $\tilde{x}_0$ is assumed to be distributed according to $\pi(x)$. To achieve the best possible accuracy for a given value of N, we will assume $C = \pi$ so that $c_{N-1} = (2N-3)\pi/(2N-1)$.

The approximate distribution of T is given by [8]

$$\begin{aligned} \Psi_N(t) &= \Pr\{T = t\} \\ &= \begin{cases} 2\hat{\pi}_{N,N-1}, & t = 0 \\ (1 - 2\hat{\pi}_{N,N-1})(1-q)q^{t-1}, & t \ge 1 \end{cases} \end{aligned}$$

where

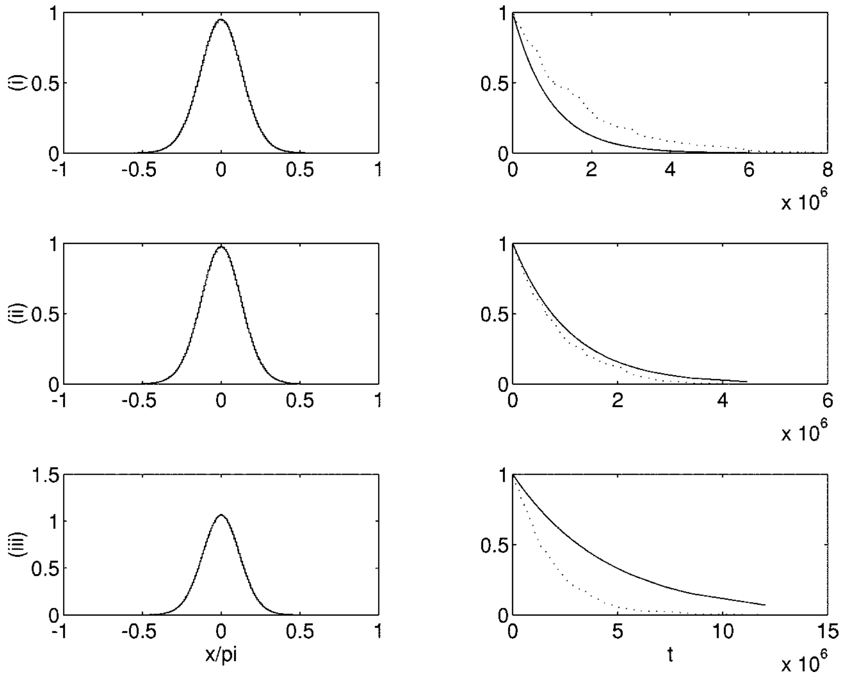
$$q = 1 - \frac{2\hat{\pi}_{N,N-1}}{1 - 2\hat{\pi}_{N,N-1}}(1 - \hat{p}_{N-1,N-1} - \hat{p}_{N-1,1-N}).$$

From this, we can show that the mean of T is approximately

$$E\{T\} = \frac{1 - 2\hat{\pi}_{N,N-1}}{1-q}.$$

IV. Examples

In these examples, we calculate $\hat{\Pi}_N$ and $\Psi_N(t)$ for three different signals and compare them with simulated approxima-



(a) (b)

Fig. 1. (a) Approximate (-) and simulated $(\cdots)$ $\pi(x)$. (b) Approximate (-) and simulated $(\cdots)$ $R(T_i)$. (i) $\sigma_u^2 = 0.02, \mathrm{SNR} = -5$, (ii) $\sigma_u^2 = 0.035, \mathrm{SNR} = -2.5$, (iii) $\sigma_u^2 = 0.05, \mathrm{SNR} = 0$.

tions. $N = 100$ and $\epsilon = 0.000\,01$ were used in the calculation of $\hat{\Pi}_N$ while $N = 50$ has been used to compute $\Psi_N(t)$. Instead of plotting $\Psi_N(t)$, the function

$$R(t) = \Pr\{T > t\} = 1 - \sum_{s=0}^{t} \Psi_N(t).$$

is considered.

The simulated approximations of Π_N are based on 1000 simulated realizations of $\tilde{x}_t,\ t = 500, 501, \cdots, 1500$, i.e., a total of 1 000 000 observations. The simulated approximations of $R(t)$ are based on initializing the phase error process according to $\hat{\Pi}_N$ 200 times and observing the first times at which $|\tilde{x}_t| > c_{N-1}$. If we denote these times by $T_i,\ i = 1, 2, \cdots, 200$, then the simulated approximation of $R(T_i)$ is given by

$$\hat{R}(T_i) = \frac{200 - i}{200}.$$

We only evaluate $R(t)$ and $\hat{R}(t)$ at $T_i,\ i = 1, 2, \cdots, 200$, to avoid having to store millions of items of data.

These functions are shown in Fig. 1 for three signals with $\sigma_u^2 = 0.02, 0.035, 0.05$ and $\mathrm{SNR} = -5, -2.5, 0$ dB, respectively. There is very little difference between the approximate and simulated $\pi(x)$ in all three cases. However, there appears to be a significant difference between $R(T_i)$ and $\hat{R}(T_i)$. This is probably due to the fact that $\hat{\pi}_{N,N-1}$ is very small and $\hat{q}$ is very close to 1, which could cause significant numerical errors. $\hat{\Pi}_N$ took two iterations to compute in all three cases. The resulting value of C was 2.6996 for both $\sigma_u^2 = 0.02$, $\mathrm{SNR} = -5$ and $\sigma_u^2 = 0.035$, $\mathrm{SNR} = -2.5$, while $\sigma_u^2 = 0.05$, $\mathrm{SNR} = 0$ resulted in $C = 2.4154$.

V. Conclusions

We have examined the properties of the phase error resulting from the EKF-PLL which tracks a random walk phase. A novel method of approximating the limit pdf of the phase error based on DMC techniques was developed. The approximate limit pdf was shown in simulations to be very accurate. Also considered was the length of time the EKF-PLL runs until it is "out of lock." The distribution of this time was derived but differed from simulation results due to possible numerical difficulties. Regardless of this, the method of calculation is still of interest.

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