TABLE II
STABILIZATION RESULTS $-\ p = 4$

Method	a_1	a_2	a_3	a_4	$d(\boldsymbol{a}, \boldsymbol{b})$
Unstable	2.100	−3.300	1.700	−0.800	—
(a)	0.266	−0.418	0.215	−0.101	3.79
(b)	1.278	−1.256	0.539	−0.183	2.57
(c)	1.035	−1.627	0.838	−0.394	2.20
(d)	1.614	−1.954	0.942	−0.383	1.67
(e)	1.539	−1.944	1.251	−0.800	1.53
(f)	2.192	−2.992	1.954	−0.794	0.41

TABLE III
STABILIZATION RESULTS $-\ p = 8$

Method	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	$d(\boldsymbol{a}, \boldsymbol{b})$
Unstable	−6.000	−12.00	−10.00	2.000	13.00	10.00	4.000	0.800	—
(a)	−0.104	−0.208	−0.173	0.035	0.225	0.173	0.069	0.014	23.4
(b)	−1.710	−0.949	0.589	1.376	1.037	0.441	0.109	0.012	22.4
(c)	−0.199	−0.398	−0.332	0.066	0.431	0.332	0.133	0.027	23.1
(d)	−2.983	−3.363	−0.831	2.417	3.206	1.847	0.607	0.097	18.5
(e)	0.407	1.921	−1.500	−0.043	1.779	−1.679	−0.686	0.800	24.4
(f)	−5.594	−11.97	−10.35	2.027	11.97	10.35	3.972	0.595	1.23

gorithm proceeds by steepest descent in the stability hypercube of the reflection space and was seen to converge efficiently to a solution under mild conditions. Besides being needed for ARMA set theoretic identification by successive projections, the method is also of interest in problems where it is needed to stabilize and render invertible a system while perturbing the least its coefficients. Thus, it finds applications in areas such as time series analysis, control, system identification, spectral estimation, and linear prediction.

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Symmetric Solutions and Eigenvalue Problems of Toeplitz Systems

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***Abstract*—Algorithms and properties of symmetric solutions of a Toeplitz system are studied and applied to an eigenvalue problem and Pisarenko's method.**

I. INTRODUCTION

The symmetric Toeplitz matrix has been studied and applied in digital signal processing extensively. A very important reason is the popular Levinson recursion. Some recent works relevant to the Levinson algorithm involve the minimum eigenvalue-eigenvector of the Toeplitz matrix [1], [2], which suggest fast algorithms for

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Pisarenko's harmonic decomposition. Also, a new algorithm [3], which solves the Yule–Walker equation more efficiently, has been reported. The crucial point of this algorithm is that the classical prediction polynomial can be recovered from two successive "singular predictor polynomials." Thus, for calculating the classical solutions, we only need about one-half of the computational cost in multiplication of the Levinson algorithm, but there is no gain in addition. Later, a lag-two recursion to calculate the odd or even symmetric (antisymmetric) solutions with less cost in addition and the same cost in multiplication was developed in [4]. However, according to the author: "Its drawback comes from the fact that the linear prediction coefficients may not be recovered in a simple way." [4].

In this correspondence, we obtain an algorithm to solve the above problem. Also, we show that the numbers of positive and negative eigenvalues associated with symmetric (antisymmetric) eigenvectors are same as the numbers of positive and negative predictor errors of symmetric (antisymmetric) filters. Based on the odd symmetric solutions and the property that all roots of symmetric and antisymmetric filters are on the unite circle, a new method for Pisarenko's decomposition is introduced. Compared with the methods suggested in [1] and [2], our method reduces the number of iteration and the computational cost in each iteration considerably. Moreover, we obtain a general method to deal with the eigenvalue problem for symmetric matrices with a high order of convergence.

II. Symmetric and Antisymmetric Prediction

It is well known that the solution

$$T_n \begin{bmatrix} 1 \\ \phi_{n1} \\ \vdots \\ \phi_{nm} \end{bmatrix} = \begin{bmatrix} r_0 & r_1 & \cdots & r_n \\ r_1 & r_0 & \cdots & r_{n-1} \\ \vdots & \vdots & \cdots & \vdots \\ r_n & r_{n-1} & \cdots & r_0 \end{bmatrix} \begin{bmatrix} 1 \\ \phi_{n1} \\ \vdots \\ \phi_{nm} \end{bmatrix} = \begin{bmatrix} \sigma_n \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{2.1}$$

has played a very important role in signal processing and time series analysis. Let $\Phi_n = [1, \phi_{n1}, \cdots, \phi_{nn}]^T$. The significance of such a solution in a prediction problem is that $\sigma_n = \Phi_n^T T_n \Phi_n = \min_a a^T T_n a$, when $T_{n-1} \geq 0$, where the minimum is taken over all $(n+1)$-dimensional vectors with first component 1. In this correspondence, we discuss the following solutions of a linear system:

$$T_{n+1} \begin{bmatrix} 1 \\ a_{n1} \\ \vdots \\ a_{nn} \\ 1 \end{bmatrix} = \begin{bmatrix} r_0 & r_1 & \cdots & r_{n+1} \\ r_1 & r_0 & \cdots & r_n \\ \vdots & \vdots & \cdots & \vdots \\ r_{n+1} & r_n & \cdots & r_0 \end{bmatrix} \begin{bmatrix} 1 \\ a_{n1} \\ \vdots \\ a_{nn} \\ 1 \end{bmatrix} = \begin{bmatrix} s_n \\ 0 \\ \vdots \\ 0 \\ s_n \end{bmatrix} \tag{2.2}$$

$$T_{n+1} \begin{bmatrix} 1 \\ b_{n1} \\ \vdots \\ b_{nn} \\ -1 \end{bmatrix} = \begin{bmatrix} r_0 & r_1 & \cdots & r_{n+1} \\ r_1 & r_0 & \cdots & r_n \\ \vdots & \vdots & \cdots & \vdots \\ r_{n+1} & r_n & \cdots & r_0 \end{bmatrix} \begin{bmatrix} 1 \\ b_{n1} \\ \vdots \\ b_{nn} \\ -1 \end{bmatrix} = \begin{bmatrix} t_n \\ 0 \\ \vdots \\ 0 \\ -t_n \end{bmatrix}. \tag{2.3}$$

If such $A_n = [1, a_{n1}, \cdots, a_{nn}, 1]^T$ and $B_n = [1, b_{n1}, \cdots, b_{nn}, -1]^T$ exist, we call them symmetric and antisymmetric solutions, respectively. In fact, if the T_{n-1} is regular, i.e., inverse of T_{n-1} exists, then A_n and B_n exist and are given by following formulas uniquely:

$$[a_{n1}, \cdots, a_{nn}]^T = -T_{n-1}^{-1}[r_1 + r_n, r_2 + r_{n-1}, \cdots, r_n + r_1]^T \tag{2.4}$$

$$[b_{n1}, \cdots, b_{nn}]^T = -T_{n-1}^{-1}[r_1 - r_n, r_2 - r_{n-1}, \cdots, r_n - r_1]^T. \tag{2.5}$$

Let J be the exchange matrix, i.e., a matrix with elements such as ones on the antidiagonal and zeros otherwise. Then, it is not difficult to show that A_n is symmetric and B_n is antisymmetric, i.e., $JA_n = A_n$ and $JB_n = -B_n$. Further, if $T_{n-1} \geq 0$, we can show that

$$2s_n = A_n^T T_{n+1} A_n = \min_V \, [1, V^T, 1] \, T_{n+1} \begin{bmatrix} 1 \\ V \\ 1 \end{bmatrix} \tag{2.6}$$

$$2t_n = B_n^T T_{n+1} B_n = \min_V \, [1, V^T, -1] \, T_{n+1} \begin{bmatrix} 1 \\ V \\ 1 \end{bmatrix} \tag{2.7}$$

where V varies over all, not necessarily being symmetric (antisymmetric), n-dimensional vectors.

Using the one-to-one mapping between an n-dimensional vector V and a polynomial $V(z) = [1, z, z^2, \cdots, z^{n-1}]\, V$, we can define the corresponding polynomials $A_n(z)$ and $B_n(z)$.

There is a lag-two recursion for A_n and B_n [4]. The polymonial version is following: let

$$\begin{aligned} \Omega_{m-2} &= r_m + r_1 + \sum_{j=1}^{m/2-1} a_{m-2,j}(r_{m-j} + r_{j+1}) \\ &\qquad \text{if } m \text{ is even} \\ &= r_m + r_1 + \sum_{j=1}^{(m-3)/2} a_{m-2,j}(r_{m-j} + r_{j+1}) \\ &\qquad + a_{m-2,\frac{m-1}{2}} \, r_{\frac{m+1}{2}}, \\ &\qquad \text{if } m \text{ is odd} \end{aligned} \tag{2.8}$$

$$c_m = \frac{s_{m-2}}{s_{m-4}}, \quad d_m = \frac{\Omega_{m-4}}{s_{m-4}} - \frac{\Omega_{m-2}}{s_{m-2}} \tag{2.9}$$

then

$$A_m(z) = (1 + d_m z + z^2) A_{m-2}(z) - c_m z^2 A_{m-4}(z). \tag{2.10}$$

And the symmetric prediction error s_m is given by

$$\begin{aligned} s_m &= r_0 + r_{m+1} + \sum_{j=1}^{m/2} a_{m,j}(r_{m+1-j} + r_j), \qquad \text{if } m \text{ is even} \\ &= r_0 + r_{m+1} + \sum_{j=1}^{(m-1)/2} a_{m,j}(r_{m+1-j} + r_j) \\ &\qquad + a_{m,(m+1)/2} \, r_{(m+1)/2}, \qquad \text{if } m \text{ is odd.} \end{aligned} \tag{2.11}$$

Started from the initial values $A_{-1}(z) = 2$, $s_{-1} = 2r_0$, $A_1(z) = 1 - 2(r_1/r_0)z + z^2$, $s_1 = r_0 + r_2 - 2(r_1^2/r_0)$, $\Omega_1 = 2r_1$, $\Omega_3 = r_1 + r_3 - 2(r_1/r_0)r_2$, we can calculate A_{2m+1} and s_{2m+1} by (2.8)–(2.11), recursively.

Now, we set up some connections between A_n and Φ_n. Let $y_m =$

$1 + \phi_{m-1,m-1}$, then [3]

$$T_{m-1}(\Phi_{m-1} + J\Phi_{m-1}) = \begin{bmatrix} \sigma_{m-1} \\ 0 \\ \vdots \\ 0 \\ \sigma_{m-1} \end{bmatrix} = y_m \begin{bmatrix} s_{m-2} \\ 0 \\ \vdots \\ 0 \\ s_{m-2} \end{bmatrix}, \quad m = 2, 3, \cdots.$$

So

$$y_m A_{m-2}(z) = \Phi_{m-1}(z) + z^{m-1}\Phi_{m-1}\left(\frac{1}{z}\right),$$
$$\sigma_{m-1} = y_m s_{m-2}, \quad m = 2, 3, \cdots. \tag{2.12}$$

On the other hand, it follows from (2.1) and (2.2) that

$$T_{m+1}\left(\begin{bmatrix} \Phi_{m-1} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ J\Phi_{m-1} \end{bmatrix} - \frac{\tau_m}{s_{m-2}}\begin{bmatrix} 0 \\ A_{m-2} \\ 0 \end{bmatrix}\right) = \begin{bmatrix} s_m \\ 0 \\ \vdots \\ 0 \\ s_m \end{bmatrix}, \quad m = 2, 3, \cdots,$$

where $\tau_m = r_m + \Sigma_{j=1}^{m-1} \phi_{m-1,j} r_{m-j}$, $\Phi_0 = 1$. Thus, let $x_m = (\tau_m / s_{m-2})$, we have

$$A_m(z) = \Phi_{m-1}(z) + z^{m+1}\Phi_{m-1}\left(\frac{1}{z}\right) - x_m z A_{m-2}(z), \quad m = 2, 3, \cdots. \tag{2.13}$$

Combining (2.12) and (2.13), we have

$$A_m(z) + (x_m - y_m z) z A_{m-2}(z) = (1 - z^2)\Phi_{m-1}(z), \quad m = 2, 3, \cdots. \tag{2.14}$$

From $x_m = (\tau_m / s_{m-2})$, (2.12), and the Levinson recursion we know that

$$\frac{x_m}{y_m} = \frac{\tau_m}{s_{m-2}\, y_m} = \frac{\tau_m}{\sigma_{m-1}} = -\phi_{mm}, \quad \sigma_m = (1 - \phi_{mm}^2)\sigma_{m-1}.$$

So, when T_m, $m = 0, 1, 2, \cdots$, are regular, we have $|(x_m/y_m)| \neq 1$. Thus, it follows from (2.14) that if $A_m(z)$ has a root 1 or -1, then $A_{m-2}(z)$ also has the same root. Since $A_{-1}(z) = 2$, $A_{2k-1}(\pm 1) \neq 0$, $k = 0, 1, 2, \cdots$, and then, substituting 1 and -1 into (2.14), we have

$$x_{2m+1} - y_{2m+1} = -\frac{A_{2m+1}(1)}{A_{2m-1}(1)},$$

$$x_{2m+1} + y_{2m+1} = \frac{A_{2m+1}(-1)}{A_{2m-1}(-1)}.$$

Let

$$\alpha_m = -\frac{A_{2m+1}(1)}{A_{2m-1}(1)}$$

and

$$\beta_m = \frac{A_{2m+1}(-1)}{A_{2m-1}(-1)}.$$

It follow from (2.10) that

$$\alpha_m = -\left(2 + d_{2m+1} + \frac{c_{2m+1}}{\alpha_{m-1}}\right),$$
$$\beta_m = 2 - d_{2m+1} - \frac{c_{2m+1}}{\beta_{m-1}},$$
$$x_{2m+1} = \frac{\alpha_m + \beta_m}{2},$$
$$y_{2m+1} = \frac{\beta_m - \alpha_m}{2} \tag{2.15a}$$

where

$$\alpha_1 = -\frac{1 + a_{31} + a_{32}/2}{1 + a_{11}/2}, \quad \beta_1 = \frac{1 - a_{31} + a_{32}/2}{1 - a_{11}/2}. \tag{2.15b}$$

Thus, we can get Φ_{2m} by the following formulas.

$$\phi_{2m,2m} = y_{2m+1} - 1, \quad \phi_{2m,0} = 1, \quad \phi_{2m,1} = a_{2m+1,1} + x_{2m+1}$$
$$\phi_{2m,k} = a_{2m+1,k} + x_{2m+1} a_{2m-1,k-1} - y_{2m+1} a_{2m-1,k-2} + \phi_{2m,k-2}, \quad k = 2, 3, \cdots, 2m - 1. \tag{2.16}$$

Φ_{2m+1} can be calculated by Levinson recursion based on Φ_{2m} and $\phi_{2m+1,2m+1} = -(x_{2m+1}/y_{2m+1})$.

Thus, for calculating an Φ_n by our method, the major computational cost comes from (2.8), (2.10), and (2.11); and then, the total cost is $(n^2/2) + O(n)$ in multiplication and $(7/8)n^2 + O(n)$ in addition. Comparing with the method in [3], we reduce the cost in addition by $(n^2/8) + O(n)$.

A very important property of the filter $\Phi_n(z)$ is this: all the roots of $\Phi_n(z)$ are outside the unit circle if $T_n > 0$. There is also a corresponding result for the $A_n(z)$ and $B_n(z)$.

Lemma 1 [5], [6]: Suppose that $T_n > 0$, then all roots of $A_n(z)$ and $B_n(z)$ must be on the unit circle. Further, for any real coefficient polynomial $P(z)$ with degree n and integer $m \geq n$, if all roots of $P(z)$ are not inside the unit circle, then all roots of both polynomials $P(z) + z^m P((1/z))$, and $P(z) - z^m P((1/z))$ are on the unit circle.

An important connection between classical prediction errors and eigenvalues of a Toeplitz system is that [2]: all the eigenvalues are nonnegative if, and only if, all prediction errors are nonnegative. It comes from the fact that the diagonal matrix diag $[\sigma_0, \sigma_1, \cdots, \sigma_{n+1}]$ and the Toeplitz matrix T_{n+1} are congruent for the Cholesky decomposition. Parallel to such a decomposition, the symmetric and antisymmetric solutions decompose T_{n+1}^{-1} [4]. Actually, let

$$V = \begin{bmatrix} 1 & 0 & \cdots & 0 & 1 \\ a_{11} & 1 & \cdots & 1 & b_{11} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ a_{n1} & 1 & \cdots & -1 & -b_{n1} \\ 1 & 0 & \cdots & 0 & -1 \end{bmatrix} = [V_+, V_-],$$

where the V_+ and V_- are the appropriate block matrices such that $JV_+ = V_+, JV_- = -V_-$. Then

$$V^T T_{n+1} V = \text{diag}\,[s_n, s_{n-2}, \cdots, t_{n-2}, t_n]. \qquad (2.17)$$

Also, we have the unitary matrix U such that [8]

$$U^T T_{n+1} U = \text{diag}\,[\lambda_1, \lambda_2, \cdots, \mu_1, \mu_2, \cdots],$$
$$U = [U_+, U_-]$$

where U_+ and U_- consist of a symmetric and antisymmetric eigenvector, respectively; and λ_k and μ_k are eigenvalues associated with symmetric and antisymmetric eigenvector, respectively. Since $V_+^T U_- = (JV_+)^T(JU_-) = -V_+^T U_-$, we know that $V_+^T U_- = 0$. Also, $V_-^T U_+ = 0$. Thus

$$\begin{aligned}\text{diag}\,&[s_n, s_{n-2}, \cdots, t_{n-2}, t_n]\\ &= V^T T_{n+1} V = V^T U \,\text{diag}\,[\lambda_1, \lambda_2, \cdots, \mu_1, \mu_2, \cdots]\, U^T V\\ &= \begin{bmatrix} V_+^T \\ V_-^T \end{bmatrix} [U_+, U_-] \begin{bmatrix} E_+ & 0 \\ 0 & E_- \end{bmatrix} \begin{bmatrix} U_+^T \\ U_{:ms}^T \end{bmatrix} [V_+, V_-]\\ &= \begin{bmatrix} V_+^T U_+ E_+ (V_+^T U_+)^T & 0 \\ 0 & V_-^T U_- E_- (V_-^T U_-)^T \end{bmatrix}\end{aligned}$$

where E_+ diag $[\lambda_1, \lambda_2, \cdots]$ and $E_- = \text{diag}\,[\mu_1, \mu_2, \cdots]$. So

$$\text{diag}\,[s_n, s_{n-2}, \cdots] = V_+^T U_+ E_+ (V_+^T U_+)^T,$$
$$\text{diag}\,[\cdots, t_{n-2}, t_n] = V_-^T U_- E_- (V_-^T U_-)^T.$$

Since $V_+^T U_+$ and $V_-^T U_-$ are regular; and diag $[s_n, s_{n-2}, \cdots]$ and E_+, diag $[\cdots, t_{n-2}, t_n]$ and E_- are congruent matrices, we have the following lemma.

Lemma 2: The symmetric prediction errors $\{s_n, s_{n-2}, \cdots\}$ and the eigenvalues associated with symmetric eigenvector $\{\lambda_1, \lambda_2, \cdots\}$, or the antisymmetric prediction errors $\{t_n, t_{n-2}, \cdots\}$ and the eigenvalues associated with antisymmetric eigenvector $\{\mu_1, \mu_2, \cdots\}$, have same numbers in positive, negative, and zero.

III. HARMONIC DECOMPOSITION

A very important problem in signal processing and time series analysis is to find the hidden period, i.e., to find the ω_j's in the following model

$$x_t = \sum_{j=1}^{p} C_j \cos(t\omega_j + \theta_j) + w_t. \qquad (3.1)$$

When the $\{w_t\}$ is an i.i.d. sequence, Pisarenko suggested a method to find ω_j's by solving the minimum eigenvalue-eigenvector problem of the covariance Toeplitz matrix. Some algorithms have been suggested to reduce the computational cost for this method [1], [2]. The basic idea is to transform the minimum eigenvalue-eigenvector problem to a classical prediction problem

$$(T_{2p} - xI)\Phi(x) = \begin{bmatrix} \sigma(x) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad (3.2)$$

where T_{2p} should be the theoretical covariance matrix $[r_{i-j}]_{i,j=0,1,\cdots,2p}$, $r_t = E\{x_t x_0\}$; and $\Phi(x)$ is the solution of such a linear system, which depends on a real variable x. In practice, the $[r_{i-j}]$ is not known and is replaced by the sample covariance matrix $[\hat{r}_{i-j}]_{i,j=0,1,\cdots,2p}$. Thus, x is the minimum eigenvalue of T_{2p} if, and only if, $\sigma(x) = 0$ and $T_{2p-1} - xI \geq 0$. Reference [1] suggests a mixed method of Rayleigh quotient and bisection to solve this problem. However, [2] argues that for $p > 1$, most of the iterations involve bisections. So, they prefer the bisection method and use the Levinson recursion to solve the equation (3.2) to do the iteration.

In this section, a more efficient algorithm is introduced to find the minimum eigenvalue λ and corresponding eigenvector based on the results in the previous section.

First, since we have to use the sample covariance matrix instead of the theoretical one in practice, there is no significance to calculate the λ to a very high precision. In fact, according to the law of iterated algorithm in probability, the precision of the $\hat{r}_t$ corresponding to the theoretical covariance r_t is $O((\log\log N/N)^{1/2})$, where N is the sample size (see, for example, [7]). However, all previous method need a high precision solution for the λ (for example, 10^{-6} suggested in [2]) since otherwise the obtained eigenfilter may not have all roots on the unit circle. To overcome this, instead of (3.2) we consider the symmetric prediction equation

$$(T_{2p} - xI)A_{2p-1}(x) = \begin{bmatrix} s_{2p-1}(x) \\ 0 \\ \vdots \\ 0 \\ s_{2p-1}(x) \end{bmatrix}. \qquad (3.3)$$

According to the Lemma 1, the polynomial $[1, z, \cdots, z^{2p}]A_{2p-1}(x)$ has all its roots on the unit circle provided that $T_{2p-1} - xI > 0$. The later condition holds if the absolute values of all reflect coefficients in (3.2), up to $2p - 2$, are smaller than one. So we only need to find an x such that $|x - \lambda| < (C/N^{1/2})$ and $|\phi_{jj}(x)| < 1, j = 1, 2, \cdots, 2p - 2$, then $A_{2p-1}(x)$ can be taken as an estimator of the desired eigenfilter of the theoretical covariance matrix. Experiences show that the estimation precision depends on the signal noise ratio (SNR) when the sample size N is given. So we take $C = (x/10\hat{r}_0)$. This can reduce the number of iterations considerably without lost estimate precision.

Secondly, the Rayleight quotient iteration is efficient only if a good initial value is available. So, we use the bisection method at the beginning. However, a modification can be performed to get some gain without much more computational cost. Since $T_{2p} - xI \geq (\lambda - x)I$ for all x, we have

$$\begin{aligned}\frac{2s_{2p-1}(x_k)}{\|A_{2p-1}(x_k)\|^2} &= \frac{A_{2p-1}(x_k)^T(T_{2p} - x_k I)A_{2p-1}(x_k)}{\|A_{2p-1}(x_k)\|^2}\\ &\geq \lambda - x_k. \end{aligned}\qquad (3.4)$$

So, in each iteration the upper bound may be set by $x_k + (2s_{2p-1}(x_k)/\|A_{2p-1}(x_k)\|^2)$.

Thirdly, it has been known that a $(2p + 1) \times (2p + 1)$ Toeplitz matrix has $p + 1$ symmetric eigenvectors and p antisymmetric eigenvectors [8]. Since the desired eigenfilter, which is given by $E(z) = \Pi(1 - 2\cos\omega_j z + z^2)$, is a symmetric one, we only need to find the minimum eigenvalue λ among the eigenvalues corresponding to the symmetric eigenvectors. However, to check if $\lambda > x$, according to the Lemma 2, we only need to verify whether all $s_{2j-1}(x), i = 1, 2, \cdots, p$, are positive. This can be performed by the recursion (2.8)-(2.11) and reduce the computational cost in each iteration to one-half of the algorithm suggested in [2].

Finally, when the matrix $(T_{2p-2} - xI)$ is regular, we have from (3.3) that

$$[a_1(x), \cdots, a_p(x), \cdots, a_1(x)]^T = -(T_{2p-2} - xI)^{-1}\vec{r} \qquad (3.5)$$

TABLE I
ESTIMATION PRECISION AND NUMBERS OF ITERATIONS[a]

			Number of Iteration			Our Method		IMSL	
ω_j's	SNR	Sample Size	Step I	Step II	Coefficient[b]	Average	S.D.	Average	S.D.
0.73	9	100	1.00	2.90	−1.4903	−1.4776	0.0107	−1.4776	0.0107
		500	1.00	3.00		−1.4794	0.0041	−1.4794	0.0041
		1000	1.00	3.00		−1.4909	0.0033	−1.4909	0.0033
	−3	100	6.03	0.00		−1.4360	0.1773	−1.4155	0.3261
		500	7.00	0.00		−1.4797	0.0704	−1.4929	0.0732
		1000	7.27	0.00		−1.4761	0.0476	−1.4772	0.0473
0.73	0	100	2.57	1.54	−0.9007	−0.7776	0.1079	−0.7716	0.1061
1.77					1.1214	1.1107	0.0753	1.1136	0.0750
		500	1.37	2.57	−0.9007	−0.9017	0.0400	−0.9019	0.0397
					1.1214	1.1226	0.0359	1.1227	0.0357
		1000	1.10	2.57	−0.9007	−0.7992	0.0255	−0.7993	0.0255
					1.1214	1.1171	0.0233	1.1172	0.0233
0.73	2	100	5.96	0.46	−1.3193	−1.3257	0.3307	−1.3477	0.3379
1.36					2.4973	2.4651	0.3405	2.4913	0.3529
1.77					−2.2709	−2.2603	0.5513	−2.2973	0.5665
		500	7.77	0.00	−1.3193	−1.3196	0.1306	−1.3300	0.1313
					2.4973	2.4942	0.1300	2.5057	0.1315
					−2.2709	−2.2707	0.2157	−2.2777	0.2173
		1000	7.42	0.00	−1.3193	−1.3060	0.0725	−1.3126	0.0721
					2.4973	2.4749	0.0794	2.4923	0.0793
					−2.2709	−2.2473	0.1237	−2.2590	0.1233

[a] The ω_j's and SNR are the same as in [2, table I]. Each model has been simulated 100 times.
[b] The coefficients of a_j, $j = 1, 2, \cdots, m$, where $\Sigma_{j=0}^{2p} a_j z^j = \Pi_{j=0}^{m} (1 - 2 \cos \omega_j z + z^2)$.
Note: "Average" and "S.D." are for the sample mean and standard deviation of the 100 estimators in each time, respectively. "IMSL" is for the estimator calculated by the routine EVESF in the software IMSL.

where $[1, a_1(x), \cdots, a_p(x), \cdots, a_1(x), 1] = A_{2p-1}(x)^T$ and $\vec{r} = [\hat{r}_1 + \hat{r}_{2p-1}, \hat{r}_2 + \hat{r}_{2p-2}, \cdots, \hat{r}_{2p-1} + \hat{r}_1]^T$. So, we can give an analytical representation of the function $s_{2p-1}(x)$ in (3.3) by following:

$$\begin{aligned} s_{2p-1}(x) &= \hat{r}_0 - x + \sum_{j=1}^{p-1} a_j(x)(\hat{r}_j + \hat{r}_{2p-j}) + a_p(x)\hat{r}_p + \hat{r}_{2p} \\ &= \hat{r}_0 - x + \hat{r}_{2p} + \tfrac{1}{2}\left[\sum_{j=1}^{b-1} a_j(x)(\hat{r}_j + \hat{r}_{2p-j}) + a_m(x)\right. \\ &\quad \left. \cdot (\hat{r}_m + \hat{r}_m) + \sum_{j=1}^{b-1} a_j(x)(\hat{r}_{2p-j} + \hat{r}_j)\right] \\ &= \hat{r}_0 - x + \hat{r}_{2p} - \tfrac{1}{2}\vec{r}^T(T_{2p-2} - xI)^{-1}\vec{r}. \end{aligned} \tag{3.6}$$

Then, the first derivative and the nth derivative of $s_{2p-1}(x)$ are given by

$$\begin{aligned} s'_{2p-1}(x) &= -1 - \frac{1}{2}\hat{r}^T(T_{2p-2} - xI)^{-2}\vec{r} \\ &= -1 - \frac{a_m(x)^2}{2} - \sum_{j=1}^{m-1} a_j(x)^2 = -\frac{1}{2}\|A_{2p-1}(x)\|^2 \end{aligned} \tag{3.7}$$

$$s_{2p-1}^{(n)}(x) = -\frac{n!}{2}\vec{r}^T(T_{2p-2} - xI)^{-n-1}\vec{r}. \tag{3.8}$$

So, when $T_{2p-2} - xI > 0$, i.e., $x < \mu$, hereafter μ is the minimum eigenvalue of T_{2p-2}, $s_{2p-1}(x)$ has negative derivatives with all rank. This means that $s_{2p-1}(x)$ is decreasing convex function in $[0, \mu]$ and has the Taylor expansion

$$\begin{aligned} s_{2p-1}(x) &= s_{2p-1}(x_0) - \frac{1}{2}\|A_{2p-1}(x_0)\|^2(x - x_0) \\ &\quad - \sum_{n=2}^{\infty} \frac{n!}{2}\vec{r}^T(T_{2p-2} - xI)^{-n-1}\vec{r}(x - x_0)^n, \\ &\quad x_0, x < \mu \end{aligned} \tag{3.9}$$

Thus, if $x_k < \mu$, $s_{2p-1}(x_k) - (1/2)\|A_{2p-1}(x_k)\|^2(x - x_k)$ is the first-order approximation of $s_{2p-1}(x)$, and the Rayleigh quotient $q_k = x_k + (2s_{2p-1}(x_k)/\|A_{2p-1}(x_k\|^2)$ is the zero of such a polynomial, i.e., the approximate value given by Newton–Raphson's method for solving the equation $s_{2p-1}(x) = 0$. Also, if there is a $x_{k'} < \mu$ such that $s_{2p-1}(x_{k'})s_{2p-1}(x_k) < 0$, then $f_k = x_k - (s_{2p-1}(x_k)/(s_{2p-1}(x_k) - s_{2p-1}(x_{k'}))(x_k - x_{k'})$ is the approximate value given by secant (*regula falsi*) method. Noting that $s_{2p-1}(x)$ is convex function in $(0, \mu)$ and (3.4), we have $f_k > \lambda > q_k$, so $(f_k + q_k/2)$ can be the approximate value of λ.

Algorithm: Initial values: let $LB_0 = 0$ and $UB_0 = \hat{r}_0 - max \{|\hat{r}_j|, j = 1, 2, \cdots, 2p\}$.

Step 1: Let $x_k = (LB_k + UB_k/2)$. Calculate $s_{2j-1}(x_k)$, $j = 0, 1, \cdots$, by the recursion (2.8)–(2.11) until some $s_{2j-1}(x_k) < 0$ or $j = p$. Let $UB_{k+1} = \min \{UB_k, x_k + (2s_{2j-1}(x_k)/\|A_{2j-1}(x_k)\|^2)\}$.

```
  If $(UB_{k+.l} - LB_k)/2 < (x_k/10\hat{r}_0 N^{1/2})$ and $|\phi_{jj}(x_k)| < 1, j = 1, \cdots, 2p - 2$, then stop, else
      If $s_{2j-1}(x_k) > 0, = j = 0, 1, \cdots, p - 1$, and $s_{2p-1}(x_k) < 0$, then $UB_k = x_k$ and go to Step 2.
      If $s_{2j-1}(x_k) > 0, j = 0, 1, \cdots, p$, then, $LB_{k+1} = x_k$.
      If there is a $j$, $s_{2j-1}(x_k) < 0$, then $LB_{k+1} = LB_k$.
  Continue Step 1.
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Step 2: Let $f_k = LB_k - s_{2p-1}(LB_k)(UB_k - LB_k)/[s_{2p-1}(UB_k) - s_{2p-1}(LB_k)]$, $q_k = x_k + 2s_{2p-1}(UB_k)/\|A_{2p-1}(x_k)\|^2$.
Let $x_{k+1} = (f_k + q_k)/2$. **If** $(q_k - f_k)/2 < (x_{k+1}/10\hat{r}_0 N^{1/2})$ and $|\phi_{2p-2,2p-2}(x_k)| < 1$, **then** stop, **else**
If $s_{2p-1}(x_{k+1}) < 0$ **then** $UB_{k+1} = x_{k+1}$, **else** $LB_{k+1} = x_{k+1}$.
Continue Step 2.

Finally, we point out that the second step can be modified to find a precise approximate value of λ for a large-scale matrix T_{2p}. Let $A'(y) = (T_{2p-2} - yI)^{-1}\vec{r}$, from (3.9) we know that

$$P(x) = s_{2p-1}(y) - \tfrac{1}{2}[\|A_{2p-1}(y)\|^2(x - y) + A_{2p-1}(y)^T A'(y)(x - y)^2 + \|A'(y)\|^2(x - y)^3] \tag{3.10}$$

is the third-order approximation of $s_{2p-1}(x)$ by Taylor expansion provided that $x, y < \mu$. So, when we find a x_k such that $\lambda < x_k < \mu$, let $y = x_k$ and x_{k+1} be the zero of $P(x)$, we can set up an iteration to approximate the λ by a forth order of convergence. The additional computational cost for calculating $A'(x_k)$, which can be obtained as $V \operatorname{diag}[s_{2p-3}^{-1}, \cdots, t_{2p-3}^{-1}] V^T \vec{r}$ by using (2.17), is only $O(p^2)$. The $\{x_k\}$ is decreasing but greater than λ, so this iteration can be performed successively. Further, for a general matrix M, the above method still works if we use the (3.2) and Cholesky factorization of M^{-1}, which can also be performed by calculating the classical predictor solutions, instead of (3.3) and the recursion given by (2.8)–(2.11). Also, the method is able to calculate the eigenvalue of a Toeplitz or symmetric matrix.

IV. SIMULATION RESULTS

The ω_j's in (3.1) and SNR are the same as in [2], and the $\{w_t\}$ in (3.1) in a pseudo-Gaussian white noise, generated by the "International Mathematical and Statistical Library" (IMSL) routine RNNOA. The sample size varies among 100, 500, and 1000. For each model, 100 sets of samples are generated and processed by this method. The sample mean and standard deviation of the 100 estimators for the frequency model $\Pi(1 - 2\cos\omega_j z + z^2)$, which is corresponding to the eigenvector of theoretical covariance matrix associated with the minimum eigenvalue, are listed on the Table I. The frequency is also estimated by the minimum eigenvector of same sample covariance matrix using the IMSL routine EVESF. The sample mean and standard deviation of these estimators are also listed on the table. Comparing the two kinds of statistics, we can see that the precision is not sacrificed. The average numbers of iterations, for step 1 and step 2 separately, are also given in the table. Comparing both the numbers of iterations and the computational cost for each iteration with that in [2, table I], we can see the efficiency of our algorithm.

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Separation of Independent Sources from Correlated Inputs

J. L. Lacoume and P. Ruiz

Abstract—**The characterization of independent stationary stochastic components (sources), is generally achieved by using the spectral matrix of partially correlated measurements, which are linearly related to the components of interest. In the general case where no assumptions are made concerning the way the sources are mixed on the measurements, the spectral matrix is not able to extract the true sources. While spectral analysis only uses second-order properties of independent stochastic sources, a procedure based on higher order analysis (fourth-order cross cumulants) is developed. This original approach leads to a complete identification of the sources.**

I. INTRODUCTION

In a noisy environment, vibrating sources rotating around the same frequency f form independent components when any of them is physically not related to each other. Because of the environment, the independent components (which will be named sources) cannot be recorded separately, but are mixed on a set of measurements. The mixture can be unknown, but is supposed linear. The measurements will be named inputs, and are, because of the mixture, partially correlated. We are here concerned with the extraction around frequency f of the independent components from the correlated inputs, i.e., with the identification of the mixture. The classical approaches of the extraction are based on a Gram–Schmidt orthogonalization of the inputs: MISO [1] or on an eigenvectors decomposition of the spectral matrix of the inputs [2]. Gram–Schmidt orthogonalization is used, in particular, in the separation of vibrating machines [1]. Eigenvectors decomposition is successfully used in underwater beamforming, assuming the plane wave hypothesis for the modelization of the sources [2], and leads to the MUSIC algorithm. Nevertheless, in the general case where no more hypothesis than the independency can be formulated, the extraction of independent components fails by only using the second-order approach. Our aim is to show that such a problem is overcome using higher order statistics (moments or cumulants). The use of third- or fourth-order statistics appeared recently on various domains of signal processing. A synthesis of this new signal characterization and of a large panel of applications are given in [3]. We

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